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Necessary conditions for the existence of conditional moments of stable random variables[☆]

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Abstract

Let (X_1, X_2) be a symmetric α-stable random vector with $0 < \alpha < 2$. Its distribution is characterized by a finite measure Γ on the unit circle called the spectral measure. It is known that if Γ satisfies some integrability condition then the conditional moment $E[|X_2|^p | X_1 = x]$ can exist for $\alpha \leqslant p < 2\alpha + 1$. The paper shows that this sufficient condition is also necessary in the cases $\alpha < p < 2\alpha + 1$ if either $0 < \alpha \leqslant 1/2$ or $1 < \alpha \leqslant 3/2$, $\alpha < p \leqslant 2$ if $1/2 < \alpha \leqslant 1$ and $\alpha < p \leqslant 4$ if $3/2 < \alpha < 2$. It also provides a sufficient and necessary condition for the existence of $E[|X_2|^\alpha | X_1 = x]$ (i.e. $p = \alpha$) for $0 < \alpha < 2$.

Keywords: Stable distributions; Bivariate stable distributions; Conditional moments; Regression

1. Introduction and statement of the main result

Let (X_1, X_2) be an α-stable, $0 < \alpha < 2$, random vector. We assume throughout the paper that the components of the vector (X_1, X_2) are not linearly dependent, i.e. (X_1, X_2) is truly two-dimensional.

It is known that $E|X_2|^p < \infty$ if and only if $p < \alpha$. However, if one considers the existence of conditional moments $E[|X_2|^p | X_1]$, then the range of p can be wider, i.e. it is possible to have all $p < 2\alpha + 1$. This is the case, for example, if X_1 and X_2 are two marginal values of a two-sided Ornstein–Uhlenbeck process (see Samorodnitsky and Taqqu, 1994, p. 252). The form of the conditional variance ($p = 2$) for $\alpha > 1/2$ is given in Cioczek-Georges and Taqqu (1994a).

In the paper Cioczek-Georges and Taqqu (1994b) (see also Cioczek-Georges and Taqqu, 1994c where the case $1 < \alpha < 2$ is studied in great detail) we give a sufficient condition for the existence of $E[|X_2|^{\alpha+\nu} | X_1 = x]$ a.e. in terms of the spectral measure

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Γ on the unit circle S_2 associated with the vector (X_1, X_2). Namely:

If $\int_{S_2} |s_1|^{-\nu}\Gamma(\mathrm{d}s) < \infty$ *for some* $0 < \nu < \alpha + 1$, *then* $E[|X_2|^{\alpha+\nu}|X_1 = x] < \infty$ *a.e.*

We conjecture that the above implication is, in fact, an equivalence. The main theorem verifies this conjecture in the case of conditional moments of order not exceeding 2 and symmetric α-stable ($S\alpha S$) random vectors (X_1, X_2).

Theorem 1.1. *Let* (X_1, X_2) *be a* $S\alpha S$, $0 < \alpha < 2$, *random vector and suppose* $0 \leqslant \nu < \alpha + 1$ *if* $\alpha \leqslant 1/2$ *or* $0 \leqslant \nu \leqslant 2 - \alpha$ *if* $1/2 < \alpha < 2$. *Then* $E[|X_2|^{\alpha+\nu}|X_1 = x] < \infty$ *for all*[1] $x \in \mathbb{R}$ *if and only if*

$$-\int_{S_2} \ln|s_1|\Gamma(\mathrm{d}s) < \infty \quad \text{if } \nu = 0,$$

$$\int_{S_2} |s_1|^{-\nu}\Gamma(\mathrm{d}s) < \infty \quad \text{if } \nu = 0. \tag{1.1}$$

This theorem contains not only the converse to the main result of our paper (1994b) in the specified range of ν, but it also provides a sufficient and necessary condition for the existence of $E[|X_2|^{\alpha}|X_1]$ ($\nu = 0$) – a case which was not covered in that paper.

Wu and Cambanis (1991) proved the above theorem for $\nu = 2 - \alpha$ and $1 < \alpha < 2$. Using the technique developed in the proof of Theorem 1.1 we are able to extend its statement to moments greater than 2 but not exceeding 4 in the case $1 < \alpha < 2$.

Theorem 1.2. *Let* (X_1, X_2) *be a* $S\alpha S$, $1 < \alpha < 2$, *random vector and suppose* $2 - \alpha < \nu < \alpha + 1$ *if* $1 < \alpha \leqslant 3/2$ *or* $2 - \alpha < \nu \leqslant 4 - \alpha$ *if* $3/2 < \alpha < 2$. *Then* $E[|X_2|^{\alpha+\nu}|X_1 = x] < \infty$ *for all* $x \in \mathbb{R}$ *if and only if*

$$\int_{S_2} |s_1|^{-\nu}\Gamma(\mathrm{d}s) < \infty. \tag{1.2}$$

Ideally, one would like to show the equivalence between the existence of conditional moments and the condition on the spectral measure for all $\nu < \alpha + 1$. Note that the two preceding theorems do so in the cases $0 < \alpha \leqslant 1/2$ and $1 < \alpha \leqslant 3/2$.

The proof of the sufficient condition in Cioczek-Georges and Taqqu (1994b) is based on Ramachandran's Theorem 5 in Ramachandran (1969). To prove the necessity of (1.1) or (1.2), we also need that theorem but in the following slightly different formulation. (Terminology: A distribution F is said to have moment of order λ if $\int_{-\infty}^{\infty} |x|^{\lambda}F(\mathrm{d}\lambda) < \infty$.)

Theorem 1.3 (cf. Ramachandran, 1969). *Let F be a distribution function and ϕ the corresponding characteristic function.*

(i) *A necessary and sufficient condition for F to have the moment of order λ, where* $0 < \lambda < 2$, *is that for some* $c > 0$

$$\int_0^c r^{-(1+\lambda)}(1 - \mathrm{Re}\,\phi(r))\mathrm{d}r < \infty.$$

[1] Since a joint density exists, it can be used to compute the conditional moments. In the $S\alpha S$ case, it is everywhere positive and therefore the conditional moments are defined "for all $x \in \mathbb{R}$."

Moreover, F has the second moment if and only if $r^{-2}(1 - \operatorname{Re}\phi(r))$ *is bounded for* $0 < r \leqslant c$.

(ii) *A necessary and sufficient condition for F, having the moment of order* $2n$, $n > 0$, *to have the moment of order* $2n + \lambda$, *where* $0 < \lambda < 2$, *is that for some* $c > 0$

$$\int_0^c r^{-(1+\lambda)}(\operatorname{Re}\phi^{(2n)}(0) - \operatorname{Re}\phi^{(2n)}(r))\mathrm{d}r < \infty.$$

Moreover, F has the moment of order $2n + 2$ *if and only if* $r^{-2}(\operatorname{Re}\phi^{(2n)}(0) - \operatorname{Re}\phi^{(2n)}(r))$ *is bounded for* $0 < r \leqslant c$.

To verify this theorem, check the proof of Theorem 5 in Ramachandran (1969). (See also the monograph by Samorodnitsky and Taqqu, 1994.)

In the proof of Theorem 1.1 we set $\phi := \phi_{X_2|x}$, i.e. the conditional characteristic function of $X_2|X_1 = x$, where (X_1, X_2) is $S\alpha S$. We represent $1 - \phi_{X_2|x}(r)$ as a sum of terms. All but one term can be easily integrated w.r.t. $r^{-(1+\alpha+\nu)}\mathrm{d}r$ over $(0,1)$, in some range of ν, whether the conditional moment of order $(\alpha + \nu)$ exists or does not. The remaining term is the crucial one because the finiteness of its integral w.r.t. $r^{-(1+\alpha+\nu)}\mathrm{d}r$ is equivalent to (1.1).

The proof of Theorem 1.2 is based on a similar idea where instead of $1 - \phi_{X_2|x}(r)$ we consider $\phi''_{X_2|x}(0) - \phi''_{X_2|x}(r)$ and use part (ii) of the above theorem.

Theorems 1.1 and 1.2 are proved in two following sections and the last section contains some technical results. We will often refer to the papers (1994b) and (1994c), that is to Cioczek-Georges and Taqqu (1994b) and Cioczek-Georges and Taqqu (1994c), respectively.

2. Proof of Theorem 1.1

In this section we present several auxilliary results which ultimately lead to Theorem 1.1. We start with some definitions and notation.

Let (X_1, X_2) be a $S\alpha S$, $0 < \alpha < 2$ random vector. Then its characteristic function equals

$$\phi(t,r) := E\exp(\mathrm{i}(tX_1 + rX_2)) = \exp\left(-\int_{S_2} |ts_1 + rs_2|^\alpha \Gamma(\mathrm{d}\mathbf{s})\right),$$

where Γ, called the spectral measure, is a finite symmetric measure on the Borel sets of the unit circle S_2 in $\mathbb{R}^2$. Let us denote the scale parameter of the random variable X_i by $\sigma_i := (\int_{S_2} |s_i|^\alpha \Gamma(\mathrm{d}\mathbf{s}))^{1/\alpha}$, $i = 1, 2$.

Since $X_1 \not\equiv 0$ (we assumed X_1 and X_2 are linearly independent) the conditional characteristic function $\phi_{X_2|x}$ of X_2 given $X_1 = x$ is of the form (cf. Samorodnitsky and Taqqu, 1991)

$$\phi_{X_2|x}(r) = \frac{1}{2\pi f(x)} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tx} \phi(t,r)\mathrm{d}t,$$

where f denotes the density of the ($S\alpha S$) random variable X_1. Note that $f(x) \neq 0$ for $x \in \mathbb{R}$ since X_1 is symmetric, so that $\phi_{X_2|x}(r)$ exists for all x.

We are going to use Ramachandran's Theorem 1.3 and, hence, we have to examine the behavior of $1 - \operatorname{Re}\phi_{X_2|x}(r)$. As in our paper (1994b), we can represent $2\pi f(x)(1 - \operatorname{Re}\phi_{X_2|x}(r))$ as the sum of the following two integrals (defined for any x and $r > 0$):

$$I_1 := \int_{-\infty}^{\infty} \cos tx\, e^{-|t|^\alpha \sigma_1^\alpha} \int_{S_2} (|ts_1 + rs_2|^\alpha - |ts_1|^\alpha)\Gamma(\mathrm{d}s)\,\mathrm{d}t$$

$$= \int_0^{\infty} \int_{S_2} \cos tx\, e^{-t^\alpha \sigma_1^\alpha} [|ts_1 + rs_2|^\alpha + |ts_1 - rs_2|^\alpha - 2|ts_1|^\alpha]\Gamma(\mathrm{d}s)\,\mathrm{d}t,$$

$$I_2 := -\int_{-\infty}^{\infty} \cos tx \Bigg[\exp\left(-\int_{S_2} |ts_1 + rs_2|^\alpha \Gamma(\mathrm{d}s)\right) - \exp\left(-\int_{S_2} |ts_1|^\alpha \Gamma(\mathrm{d}s)\right)$$

$$+ \exp\left(-\int_{S_2} |ts_1|^\alpha \Gamma(\mathrm{d}s)\right)\int_{S_2} (|ts_1 + rs_2|^\alpha - |ts_1|^\alpha)\Gamma(\mathrm{d}s)\Bigg]\mathrm{d}t.$$

(In fact, the above I_1, I_2 differ from those defined in Proposition 2.1 of (1994b) only by the constant $2\pi f(x)$.)

In the following, however, we need a different representation for I_1 which can be obtained by integration by parts (cf. formula (2.3) in (1994b)) under the condition

$$\Gamma((s_1, s_2) = (0,1)) = 0. \tag{2.1}$$

This is implied by (1.1). If (1.1) is not assumed, the next lemma shows that (2.1) is always satisfied in the cases we consider here.

Lemma 2.1. *If* $E[|X_2|^\lambda | X_1 = x] < \infty$ *for all* $x \in \mathbb{R}$ *with* $\lambda \geqslant \alpha$ *then* (2.1) *holds.*

Proof. Suppose that (2.1) fails, where Γ is the spectral measure of the vector (X_1, X_2), i.e. $\Gamma((s_1, s_2) = (0,1)) > 0$. Then Γ', defined by $\Gamma'(A) = \Gamma(A \cap \{(0,1),(0,-1)\}^c)$, is another spectral measure. Let (Y_1, Y_2) be a $S\alpha S$ vector with the spectral measure Γ' and let Y_3 be a $S\alpha S$ random variable independent of (Y_1, Y_2), whose scale parameter σ_3 satisfies $\sigma_3^\alpha = \Gamma((s_1, s_2) = (0,1)) + \Gamma((s_1, s_2) = (0,-1))$. It is easy to see that the vectors (X_1, X_2) and $(Y_1, Y_2 + Y_3)$ are identically distributed. Moreover, $E[|X_2|^\alpha | X_1 = x] = E[|Y_2 + Y_3|^\alpha | Y_1 = x] = \infty$ by the independence of Y_3 and (Y_1, Y_2) and by Fubini's Theorem. Hence, the lemma holds ad absurdum. □

We may therefore integrate I_1 by parts without paying attention to the point $(0,1)$. The integration yields

$$I_1 = \int_0^{\infty} \int_{S_2} (x \sin tx\, e^{-t^\alpha \sigma_1^\alpha} + \cos tx\, e^{-t^\alpha \sigma_1^\alpha} \alpha \sigma_1^\alpha t^{\alpha-1})$$

$$\times \frac{1}{\alpha+1}\left[\left(t + \left|\frac{rs_2}{s_1}\right|\right)^{\alpha+1} + \left(t - \left|\frac{rs_2}{s_1}\right|\right)^{\langle\alpha+1\rangle} - 2t^{\alpha+1}\right]|s_1|^\alpha \Gamma(\mathrm{d}s)\,\mathrm{d}t$$

$$= \int_0^r (\ldots)\mathrm{d}t + \int_r^{\infty} (\ldots)\mathrm{d}t =: I_{11} + I_{12}.$$

This is the crucial representation for the equality

$$2\pi f(x)(1 - \operatorname{Re}\phi_{X_2|x}(r)) = I_1 + I_2 = I_{11} + I_{12} + I_2.$$

We shall use I_{11} and I_{12} in the case $v = 0$.

Lemma 2.2. *The following integrals are finite for* $c > 0$:

$$\int_c^\infty \frac{|I_1|}{r^{\alpha+v+1}}\mathrm{d}r < \infty \quad \text{for } v > 0. \tag{2.2}$$

$$\int_0^c \frac{|I_{11}|}{r^{\alpha+1}}\mathrm{d}r < \infty, \tag{2.3}$$

$$\int_c^\infty \frac{|I_{12}|}{r^{\alpha+1}}\mathrm{d}r < \infty. \tag{2.4}$$

Proof. By Lemma 4.2 we have $r, t > 0$,

$$\begin{aligned}
&\Bigg| \int_{S_2} (x \sin tx\, \mathrm{e}^{-t^\alpha \sigma_1^\alpha} + \cos tx\, \mathrm{e}^{-t^\alpha\sigma_1^\alpha}\alpha\sigma_1^\alpha t^{\alpha-1}) \\
&\quad \times \frac{1}{\alpha+1}\left[\left(t + \left|\frac{rs_2}{s_1}\right|\right)^{\alpha+1} + \left(t - \left|\frac{rs_2}{s_1}\right|\right)^{\langle\alpha+1\rangle} - 2t^{\alpha+1}\right]|s_1|^\alpha \Gamma(\mathrm{d}\mathbf{s})\Bigg| \\
&\quad \leqslant \text{const}\, \mathrm{e}^{-t^\alpha\sigma_1^\alpha}(1 + t^{\alpha-1}) \int_{S_2} \left|\frac{rs_2}{s_1}\right|^{\alpha+1} \frac{t}{|(rs_2/s_1)|} \cdot |s_1|^\alpha \Gamma(\mathrm{d}\mathbf{s}) \\
&\quad \leqslant \text{const}\, r^\alpha \mathrm{e}^{-t^\alpha\sigma_1^\alpha}(t + t^\alpha).
\end{aligned} \tag{2.5}$$

Now (2.2) holds since for $v > 0$, by (2.5),

$$\begin{aligned}
\int_c^\infty \frac{|I_1|}{r^{\alpha+v+1}}\mathrm{d}r &\leqslant \text{const} \int_c^\infty \frac{r^\alpha}{r^{\alpha+v+1}} \int_0^\infty \mathrm{e}^{-t^\alpha\sigma_1^\alpha}(t + t^\alpha)\mathrm{d}t\,\mathrm{d}r \\
&\leqslant \text{const} \int_c^\infty \frac{1}{r^{v+1}}\mathrm{d}r < \infty.
\end{aligned}$$

Similarly, by (2.5),

$$\int_0^c \frac{|I_{11}|}{r^{\alpha+1}}\mathrm{d}r \leqslant \text{const} \int_0^c \frac{1}{r} \int_0^r (t + t^\alpha)\mathrm{d}t\,\mathrm{d}r \leqslant \text{const},$$

and

$$\begin{aligned}
\int_c^\infty \frac{|I_{12}|}{r^{\alpha+1}}\mathrm{d}r &\leqslant \text{const} \int_c^\infty \frac{1}{r} \int_r^\infty \mathrm{e}^{-r^\alpha\sigma_1^\alpha/2}\mathrm{e}^{-t^\alpha\sigma_1^\alpha/2}(t + t^\alpha)\mathrm{d}t\,\mathrm{d}r \\
&\leqslant \text{const} \int_c^\infty \frac{\mathrm{e}^{-r^\alpha\sigma_1^\alpha/2}}{r}\mathrm{d}r \int_0^\infty \mathrm{e}^{-t^\alpha\sigma_1^\alpha/2}(t + t^\alpha)\mathrm{d}t \leqslant \text{const},
\end{aligned}$$

so that (2.3) and (2.4) hold. $\square$

Lemma 2.3. *For $0 \leqslant v < \alpha$ if $\alpha \leqslant 1$, and $0 \leqslant v < 2 - \alpha$ if $1 < \alpha < 2$,*

$$\int_0^1 \frac{|I_2|}{r^{\alpha+v+1}} \mathrm{d}r < \infty . \tag{2.6}$$

Proof. We can bound I_2 as we did in Propositions 2.1 of (1994b) by using the inequality $|\mathrm{e}^{-x} - \mathrm{e}^{-y}(x-y)| \leqslant \mathrm{e}^{-y}\mathrm{e}^{|x-y|}(x-y)^2/2$ and Lemma 3.1 of Samorodnitsky and Taqqu (1991), but now, of course, we do not assume (1.1). Hence, for $0 < r < 1$,

$$\begin{aligned} |I_2| &\leqslant \frac{1}{2}\int_{-\infty}^{\infty} \exp(-|t|^\alpha \sigma_1^\alpha) \exp\left|\int_{S_2} (|ts_1 + rs_2|^\alpha - |ts_1|^\alpha)\Gamma(\mathrm{d}s)\right| \\ &\quad \times \left(\int_{S_2} (|ts_1 + rs_2|^\alpha - |ts_1|^\alpha)\Gamma(\mathrm{d}s)\right)^2 \mathrm{d}t \\ &\leqslant \begin{cases} \frac{1}{2}\int_{-\infty}^{\infty} \exp\{-|t|^\alpha\sigma_1^\alpha + r^\alpha\sigma_2^\alpha\}(r^\alpha\sigma_2^\alpha)^2\,\mathrm{d}t \\ \frac{1}{2}\int_{-\infty}^{\infty} \exp\{-|t|^\alpha\sigma_1^\alpha + \alpha\Gamma(S_2)(r^\alpha + r|t|^{\alpha-1})\}(\alpha\Gamma(S_2)(r^\alpha + r|t|^{\alpha-1}))^2\,\mathrm{d}t \end{cases} \\ &\leqslant \begin{cases} \text{const}\, r^{2\alpha} & \text{if } \alpha \leqslant 1, \\ \text{const}\, r^2 & \text{if } 1 < \alpha < 2. \end{cases} \end{aligned} \tag{2.7}$$

Unless stated explicitly, here and in the future const denotes a finite positive constant, which may change from one expression to another and is independent from r (it depends however on Γ, α or x).

The above inequalities clearly imply (2.6) in the specified range of v. □

Now, we present several statements obtained under the assumption that $|I_2|$ is properly bounded.

Corollary 2.1. *Assume that $0 \leqslant v < \min(\alpha+1, 2-\alpha)$ and that (2.6) holds for this v. Then $E[|X_2|^{\alpha+v}|X_1 = x] < \infty$ if and only if*

$$\int_0^\infty \frac{|I_1|}{r^{\alpha+v+1}} \mathrm{d}r < \infty \quad \textit{in the case } v > 0, \tag{2.8}$$

and if and only if

$$\int_0^\infty \frac{|I_{12}|}{r^{\alpha+1}} \mathrm{d}r < \infty \quad \textit{in the case } v = 0. \tag{2.9}$$

Proof. This is an obvious corollary from Ramachandran's Theorem 1.3 with $c \leqslant 1$ and Lemma 2.2. □

Proposition 2.1. *Assume that $0 \leqslant v < \min(\alpha+1, 2-\alpha)$ and that (2.6) holds for this v. Then $E[|X_2|^{\alpha+v}|X_1 = 0] < \infty$ implies (1.1).*

Proof. By Corollary 2.1 either (2.8) or (2.9) holds with $x = 0$, i.e.

$$\int_0^\infty \frac{1}{r^{\alpha+\nu+1}} \left| \int_{ar}^\infty \int_{S_2} e^{-t^\alpha \sigma_1^\alpha} t^{\alpha-1} \left[\left(t + \left|\frac{rs_2}{s_1}\right| \right)^{\alpha+1} + \left(t - \left|\frac{rs_2}{s_1}\right| \right)^{\langle\alpha+1\rangle} - 2t^{\alpha+1} \right] |s_1|^\alpha \Gamma(\mathrm{d}s)\,\mathrm{d}t \right| \mathrm{d}r < \infty,$$

where $a = 0$ if $\nu > 0$ or $a = 1$ if $\nu = 0$.

Note that by Lemma 4.3 the integrand of the inner integral is nonnegative, so that we can drop absolute value sign, make change of variables $t := tr$, and change order of integration. We have

$$\infty > \int_a^\infty \int_{S_2} t^{\alpha-1} \left[\left(t + \left|\frac{s_2}{s_1}\right| \right)^{\alpha+1} + \left(t - \left|\frac{s_2}{s_1}\right| \right)^{\langle\alpha+1\rangle} - 2t^{\alpha+1} \right]$$

$$\times |s_1|^\alpha \Gamma(\mathrm{d}s) \int_0^\infty e^{-r^\alpha t^\alpha \sigma_1^\alpha} r^{\alpha-\nu}\,\mathrm{d}r\,\mathrm{d}t$$

$$= \frac{\Gamma((\alpha-\nu+1)/\alpha)}{\alpha\sigma_1^{\alpha-\nu+1}} \int_a^\infty \int_{S_2} t^{\nu-2} \left[\left(t + \left|\frac{s_2}{s_1}\right| \right)^{\alpha+1} + \left(t - \left|\frac{s_2}{s_1}\right| \right)^{\langle\alpha+1\rangle} - 2t^{\alpha+1} \right] |s_1|^\alpha \Gamma(\mathrm{d}s)\,\mathrm{d}t$$

$$= \mathrm{const} \int_{S_2} |s_2|^{\alpha+\nu} |s_1|^{-\nu} \times \int_{a|s_1/s_2|}^\infty t^{\nu-2} [(t+1)^{\alpha+1} + (t-1)^{\langle\alpha+1\rangle} - 2t^{\alpha+1}]\,\mathrm{d}t\, I[s_2 \neq 0]\,\Gamma(\mathrm{d}s), \quad (2.01)$$

where we used (4.1) in the first equality and made change of variables $t := t|s_2/s_1|$ in the second one.

If $\nu > 0$ ($a = 0$) then $\int_0^\infty t^{\nu-2}[(t+1)^{\alpha+1} + (t-1)^{\langle\alpha+1\rangle} - 2t^{\alpha+1}]\,\mathrm{d}t$ is positive and finite (Lemma 4.2), and the above inequality implies

$$\int_{S_2} |s_2|^{\alpha+\nu} |s_1|^{-\nu} \Gamma(\mathrm{d}s) < \infty,$$

which is equivalent to (1.1) for $\nu > 0$.

If $\nu = 0$ ($a = 1$) we want to use (4.2) to bound the integral (2.10) from below. We may do this only under the assumption $\Gamma(A) > 0$, where $A := \{(s_1, s_2) \in S_2\colon |s_1|/|s_2| < c(\alpha+1)\}$ and $c(\alpha+1)$ is defined in Lemma 4.3. Notice, however, that if this assumption does not hold then Γ has no mass in the open neighborhood of $(s_1, s_2) = (0, 1)$ and, clearly, $-\int_{S_2} \ln|s_1| \Gamma(\mathrm{d}s) < \infty$. When $\Gamma(A) > 0$, we get

$$\infty > \int_A |s_2|^\alpha \int_{|s_1/s_2|}^{c(\alpha+1)} t^{-1}\,\mathrm{d}t\,\Gamma(\mathrm{d}s) \geqslant -\int_A |s_2|^\alpha \ln\left|\frac{s_1}{s_2}\right| \Gamma(\mathrm{d}s) - \text{const}$$

$$\geqslant \text{const}\left(-\int_A \ln|s_1|\Gamma(\mathrm{d}s)\right) - \text{const},$$

since $c(\alpha+1) < 1/2$ and $|s_2|$ is bounded from below by a positive number for $(s_1, s_2) \in A$. Thus $-\int_A \ln|s_1|\Gamma(\mathrm{d}s) < \infty$, which implies (1.1) for $v = 0$. □

Proposition 2.2. *Assume* $1/2 < \alpha < 2$ *and* $|I_2| \leqslant \text{const}\, r^2$ *for* $0 < r < 1$. *Then* $E[X_2^2 | X_1 = 0] < \infty$ *implies* $\int_{S_2} |s_1|^{\alpha-2}\Gamma(\mathrm{d}s) < \infty$.

Proof. Ramachandran's Theorem 1.3 implies that for $x = 0$

$$\frac{|I_1|}{r^2} \leqslant \text{const}$$

for sufficiently small $r > 0$. But by Lemma 4.3 the integrand of I_1 for $x = 0$ is nonnegative. Proceeding as in Wu and Cambanis (1991) we use Fatou's Lemma to get

$$\infty > \liminf_{r \downarrow 0} \frac{I_1}{r^2}$$

$$\geqslant \frac{\alpha\sigma_1^\alpha}{\alpha+1} \int_0^\infty \int_{S_2} \mathrm{e}^{-t^\alpha\sigma_1^\alpha} t^{\alpha-1} \lim_{r\downarrow 0} \frac{1}{r^2}\left[\left(t + \left|\frac{rs_2}{s_1}\right|\right)^{\alpha+1} + \left(t - \left|\frac{rs_2}{s_1}\right|\right)^{\langle\alpha+1\rangle} - 2t^{\alpha+1}\right]$$

$$\times |s_1|^\alpha I[s_2 \neq 0]\Gamma(\mathrm{d}s)\,\mathrm{d}t$$

$$= \alpha^2\sigma_1^\alpha \int_0^\infty \int_{S_2} \mathrm{e}^{-t^\alpha\sigma_1^\alpha} t^{2\alpha-2}\left(\frac{s_2}{s_1}\right)^2 |s_1|^\alpha \Gamma(\mathrm{d}s)\,\mathrm{d}t$$

$$= \alpha^2\sigma_1^\alpha \int_0^\infty \mathrm{e}^{-t^\alpha\sigma_1^\alpha} t^{2\alpha-2}\,\mathrm{d}t \int_{S_2} s_2^2 |s_1|^{\alpha-2}\Gamma(\mathrm{d}s).$$

Since the integral with respect to t is finite and positive, we have $\int_{S_2} s_2^2|s_1|^{\alpha-2}\Gamma(\mathrm{d}s) < \infty$ which implies $\int_{S_2}|s_1|^{\alpha-2}\Gamma(\mathrm{d}s) < \infty$. □

The next proposition proves sufficiency of (1.1) in the case $v = 0$.

Proposition 2.3. *If* $-\int_{S_2} \ln|s_1|\Gamma(\mathrm{d}s) < \infty$ *then* $E[|X_2|^\alpha | X_1 = x] < \infty$ *for all* $x \in \mathbb{R}$.

Proof. Under the above assumption Γ has no mass at $(s_1, s_2) = (0, 1)$. Corollary 2.1 requires us to prove (2.9) since (2.6) always holds with $v = 0$. (2.9) is implied by the following inequalities:

$$\int_0^\infty \frac{|I_{12}|}{r^{\alpha+1}}\,\mathrm{d}r \leqslant \text{const} \int_0^\infty \frac{1}{r^{\alpha+1}} \int_r^\infty \int_{S_2} \mathrm{e}^{-t^\alpha\sigma_1^\alpha}(1 + t^{\alpha-1})$$

$$\times\left[\left(t + \left|\frac{rs_2}{s_1}\right|\right)^{\alpha+1} + \left(t - \left|\frac{rs_2}{s_1}\right|\right)^{\langle\alpha+1\rangle} - 2t^{\alpha+1}\right]|s_1|^\alpha\Gamma(\mathrm{d}s)\,\mathrm{d}t\,\mathrm{d}r$$

$$= \text{const} \int_1^\infty \int_{S_2} \left[\left(t + \left| \frac{s_2}{s_1} \right| \right)^{\alpha+1} + \left(t - \left| \frac{s_2}{s_1} \right| \right)^{\langle \alpha+1 \rangle} - 2t^{\alpha+1} \right] |s_1|^\alpha$$

$$\times \int_0^\infty \mathrm{e}^{-r^\alpha t^\alpha \sigma_1^\alpha} (r + t^{\alpha-1} r^\alpha) \mathrm{d}r \, \Gamma(\mathrm{d}s) \mathrm{d}t$$

$$= \text{const} \int_1^\infty \int_{S_2} t^{-2} \left[\left(t + \left| \frac{s_2}{s_1} \right| \right)^{\alpha+1} + \left(t - \left| \frac{s_2}{s_1} \right| \right)^{\langle \alpha+1 \rangle} - 2t^{\alpha+1} \right] |s_1|^\alpha \Gamma(\mathrm{d}s) \mathrm{d}t$$

$$= \text{const} \int_{S_2} \int_{|s_1/s_2|}^\infty t^{-2} [(t+1)^{\alpha+1} + (t-1)^{\langle \alpha+1 \rangle} - 2t^{\alpha+1}] |s_2|^\alpha \mathrm{d}t \, \Gamma(\mathrm{d}s)$$

$$\leqslant \text{const} \left[\int_{S_2} \int_{|s_1/s_2|}^1 t^{-1} \mathrm{d}t \, I \left[\left| \frac{s_1}{s_2} \right| \leqslant 1 \right] \Gamma(\mathrm{d}s) + \int_{S_2} \int_1^\infty t^{\alpha-3} \mathrm{d}t \, \Gamma(\mathrm{d}s) \right]$$

$$\leqslant \text{const} \left(- \int_{S_2} \ln |s_1| \Gamma(\mathrm{d}s) \right) + \text{const} < \infty,$$

where we made usual changes of variables and used (4.1) and Lemma 4.2. □

Now we are ready to conclude the proof of Theorem 1.1.

Proof of Theorem 1.1. The sufficiency of (1.1) for the existence of the conditional moment $E[X_2|^{\alpha+\nu}|X_1 = x]$ for $\nu > 0$ is stated in Theorem 1.2 of (1994b) and Proposition 2.3 proves it for $\nu = 0$. The fact that this condition is also necessary follows from the preceeding lemmas and propositions and the following considerations.

Assume that $E[X_2|^{\alpha+\nu}|X_1 = 0] < \infty$ for some ν in the considered range.

First note that if $0 \leqslant \nu < \alpha$, $\alpha < 1$, or $0 \leqslant \nu < 2 - \alpha$, $1 \leqslant \alpha < 2$, then Lemma 2.3 shows that (2.6) holds and, hence, by Proposition 2.1 condition (1.1) is satisfied. Similarly, if $\nu = 2 - \alpha$ and $1 \leqslant \alpha < 2$ then (2.7) and Proposition 2.2 imply (1.1).

Now consider $\alpha \leqslant \nu$ and $0 < \alpha < 1$. In Proposition 2.1 of (1994b) we have shown that if (1.1) is satisfied with some $\nu' > 0$ then for $0 < r < 1$

$$|I_2| \leqslant \begin{cases} \text{const}\, r^{2\alpha+2\nu'} & \text{if } \alpha \leqslant 1/2,\ \nu' < 1/2, \text{ or } \alpha > 1/2,\ \nu' < 1 - \alpha, \\ \text{const}\, r^{2\alpha+1-\varepsilon},\ 0 < \varepsilon < 1, & \text{if } \alpha \leqslant 1/2,\ \nu' = 1/2, \text{ or } \alpha = 1/2,\ \nu' > 1/2, \\ \text{const}\, r^{2\alpha+1} & \text{if } \alpha < 1/2,\ \nu' > 1/2, \\ \text{const}\, r^2 & \text{if } \alpha > 1/2,\ \nu' \geqslant 1 - \alpha. \end{cases} \tag{2.11}$$

We now present the crucial "looping argument." Since we assume $E[|X_2|^{\alpha+\nu}|X_1 = 0] < \infty$ with some $\nu \geqslant \alpha$ it is also true that $E[|X_2|^{\alpha+\nu'}|X_1 = 0] < \infty$ for all $\nu' < \alpha$. Then, by the part of the theorem which we have already proved, condition (1.1) holds with any $0 < \nu' < \alpha$ and we can use (2.11) instead of (2.7) as a bound for I_2. Depending on the value of α we get either $|I_2| \leqslant \text{const}\, r^2$ if $\alpha > 1/2$ or $|I_2| \leqslant \text{const}\, r^{2\alpha+2\nu'} \leqslant \text{const}\, r^{4\alpha-\varepsilon}$, $0 < \varepsilon < 2\alpha$, if $\alpha \leqslant 1/2$. In the first case, since (2.6) holds with any $\alpha \leqslant \nu < 2 - \alpha$, either Proposition 2.1 or Proposition 2.2 implies (1.1) for ν. In the second case ($\alpha \leqslant 1/2$), (2.6) holds with any $\nu'' < 3\alpha$, and hence (1.1) is satisfied for all $\nu'' < 3\alpha$ and $\nu'' \leqslant \nu$. The theorem is now proved if $\nu < 3\alpha$. If $\nu \geqslant 3\alpha$, we have to repeat the described procedure a finite number of times, i.e. until we get $|I_2| \leqslant \text{const}\, r^{2\alpha+1}$.

(Notice, for example, that $3\alpha = \alpha + 1$ for $\alpha = 1/2$, so if $v \geq 3\alpha$ then it must be $\alpha < 1/2$.) □

3. Proof of Theorem 1.2

We may assume that $E[X_2^2 | X_1 = x] < \infty$ for all $x \in \mathbb{R}$ and/or $\int_{S_2} |s_1|^{\alpha-2} \Gamma(\mathrm{d}s) < \infty$ since we consider $v > 2 - \alpha$. Under this assumption the second derivative of the characteristic function $\phi_{X_2|x}$ exists and a form of its real part $\mathrm{Re}\, \phi''_{X_2|x}(r)$ is given in Proposition 2.2 of (1994c). (In fact, in the statement of that proposition we assumed that $\int_{S_2} |s_1|^{-v} \Gamma(\mathrm{d}s) < \infty$ for some $v > 2 - \alpha$, but the proof uses only $v = 2 - \alpha$). Moreover, in the proof of Proposition 2.3 of (1994c) we established the representation

$$\mathrm{Re}\, \phi''_{X_2|x}(0) - \mathrm{Re}\, \phi''_{X|x}(r) = I_1 + I_2 + I_3,$$

where I_i, $i = 1, 2, 3$, are defined as follows:

$$I_1 := \frac{1}{2\pi f(x)} \int_{-\infty}^{\infty} \cos tx (\phi(t,0) - \phi(t,r)) \left[\left(\alpha \int_{S_2} (ts_1 + rs_2)^{\langle \alpha - 1 \rangle} s_2 \Gamma(\mathrm{d}s) \right)^2 - \alpha(\alpha-1) \int_{S_2} |ts_1 + rs_2|^{\alpha-2} s_2^2 \Gamma(\mathrm{d}s) \right] \mathrm{d}t,$$

$$I_2 := \frac{-\alpha^2}{2\pi f(x)} \int_{-\infty}^{\infty} \cos tx \phi(t,0) \left[\left(\int_{S_2} (ts_1 + rs_2)^{\langle \alpha - 1 \rangle} s_2 \Gamma(\mathrm{d}s) \right)^2 - \left(\int_{S_2} (ts_1)^{\langle \alpha - 1 \rangle} s_2 \Gamma(\mathrm{d}s) \right)^2 \right] \mathrm{d}t,$$

$$\begin{aligned} I_3 &:= \frac{\alpha(\alpha-1)}{2\pi f(x)} \int_{-\infty}^{\infty} \cos tx \phi(t,0) \int_{S_2} (|ts_1 + rs_2|^{\alpha-2} - |ts_1|^{\alpha-2}) s_2^2 \Gamma(\mathrm{d}s) \mathrm{d}t \\ &= \frac{\alpha}{2\pi f(x)} \int_0^{\infty} \int_{S_2} (x \sin tx \, \mathrm{e}^{-t^\alpha \sigma_1^\alpha} + \cos tx \, \mathrm{e}^{-t^\alpha \sigma_1^\alpha} \alpha t^{\alpha-1} \sigma_1^\alpha) \\ &\quad \times \left[\left(t + \left| \frac{rs_2}{s_1} \right| \right)^{\alpha-1} + \left(t - \left| \frac{rs_2}{s_1} \right| \right)^{\langle \alpha-1 \rangle} - 2t^{\alpha-1} \right] s_2^2 |s_1|^{\alpha-2} \Gamma(\mathrm{d}s) \mathrm{d}t. \end{aligned}$$

The idea of the proof of Theorem 1.2 is the same as that of Theorem 1.1. We are going to use Ramachandran's Theorem 1.3, part (ii) with $n = 1$. The terms I_1 and I_2 can be bounded by appropriate powers of r, $0 < r < 1$, independently of v, at least in some range of v. The term I_3 is the crucial one and the finiteness of its integral w.r.t. $r^{\alpha+v-1} \mathrm{d}r$ implies (1.2). We follow the steps of Section 2 and precede the formal proof of Theorem 1.2 with some auxiliary results.

Lemma 3.1. *For $v > 2 - \alpha$ and $c > 0$*

$$\int_c^{\infty} \frac{|I_3|}{r^{\alpha+v-1}} \mathrm{d}r < \infty.$$

Proof. Using Lemma 4.2 we obtain

$$|I_3| \leqslant \text{const} \int_0^\infty \int_{S_2} e^{-t^\alpha \sigma_1^\alpha}(1 + t^{\alpha-1})t^{\alpha-1} s_2^2 |s_1|^{\alpha-2} \Gamma(ds)dt \leqslant \text{const},$$

which implies the statement of the lemma. □

Corollary 3.1. *Assume that* $2 - \alpha < v < \min(\alpha + 1, 4 - \alpha)$ *and*

$$\int_0^1 \frac{|I_i|}{r^{\alpha+v-1}} dr < \infty, \quad i = 1,2, \tag{3.1}$$

for this v. *Then* $E[|X_2|^{\alpha+v}|X_1 = x] < \infty$ *for* $x \in \mathbb{R}$ *if and only if*

$$\int_0^\infty \frac{|I_3|}{r^{\alpha+v-1}} dr < \infty. \tag{3.2}$$

Proof. The corollary immediately follows from Ramachandran's Theorem 1.3 and the previous lemma. □

Proposition 3.1. *Assume that* $2 - \alpha < v < \min(\alpha + 1, 4 - \alpha)$ *and* (3.1) *holds for this* v. *Then* $E[|X_2|^{\alpha+v}|X_1 = 0] < \infty$ *implies* (1.2).

Proof. Corollary 3.1 implies that (3.2) holds with $x = 0$, i.e.

$$\int_0^\infty \frac{1}{r^{\alpha+v-1}} \left| \int_0^\infty \int_{S_2} e^{-t^\alpha \sigma_1^\alpha} t^{\alpha-1} \left[\left(t + \left|\frac{rs_2}{s_1}\right| \right)^{\alpha-1} \right.\right.$$
$$\left.\left. + \left(t - \left|\frac{rs_2}{s_1}\right| \right)^{\langle\alpha-1\rangle} - 2t^{\alpha-1} \right] s_2^2 |s_1|^{\alpha-2} \Gamma(ds)dt \right| dr < \infty.$$

By Lemma 4.3 the integrand of the inner integral is nonpositive. Hence, we can change its sign and drop the absolute value sign. Then, changing order of integration, using substitution and (4.1), we obtain

$$\infty > \int_0^\infty \int_{S_2} t^{\alpha-1} \left[2t^{\alpha-1} - \left(t + \left|\frac{s_2}{s_1}\right| \right)^{\alpha-1} - \left(t - \left|\frac{s_2}{s_1}\right| \right)^{\langle\alpha-1\rangle} \right] s_2^2 |s_1|^{\alpha-2}$$
$$\times \int_0^\infty e^{-r^\alpha t^\alpha \sigma_1^\alpha} r^{\alpha-v} dr\, \Gamma(ds)dt$$
$$= \text{const} \int_0^\infty \int_{S_2} t^{v-2} \left[2t^{\alpha-1} - \left(t + \left|\frac{s_2}{s_1}\right| \right)^{\alpha-1} \right.$$
$$\left. - \left(t - \left|\frac{s_2}{s_1}\right| \right)^{\langle\alpha-1\rangle} \right] s_2^2 |s_1|^{\alpha-2} \Gamma(ds)dt$$
$$= \text{const} \int_{S_2} |s_2|^{\alpha+v} |s_1|^{-v} \Gamma(ds) \int_0^\infty t^{v-2} [2t^{\alpha-1} - (t+1)^{\alpha-1} - (t-1)^{\langle\alpha-1\rangle}] dt.$$

Since the integral w.r.t. t is finite (Lemma 4.2) and positive, condition (1.2) holds. □

Proposition 3.2. *Assume that* $3/2 < \alpha < 1$ *and* $|I_i| \leqslant \text{const}\, r^2$, $0 < r < 1$, $i = 1,2$. *Then* $E[X_2^4 | X_1 = 0] < \infty$ *implies* (1.2) *with* $v = 4 - \alpha$.

Proof. Ramachandran's Theorem 1.3 implies that for $x = 0$

$$0 \leqslant \liminf_{r \downarrow 0} \frac{-I_3}{r^2} < \infty.$$

Now Fatou's Lemma implies (1.2) similarly as in the proof of Proposition 2.2. □

Proof of Theorem 1.2. We only have to prove necessity of (1.2) for the existence of $E[|X_2|^{\alpha+v}|X_1 = 0]$ in the specified range of v, since sufficiency was shown in (1994c). The necessity will follow from Proposition 3.1 or Proposition 3.2 if we find proper bounds for I_1 and I_2.

Let us suppose that $E[|X_2|^{\alpha+v}|X_1 = 0] < \infty$ for some $v > 2 - \alpha$.

In the proof of Proposition 2.3 in (1994c) we have shown that if (1.2) holds with some v' then for $0 < r < 1$,

$$|I_1| \leqslant \begin{cases} \text{const}\, r & \text{if } \alpha \leqslant 3/2,\ v' = 2-\alpha, \\ \text{const}\, r^{2\alpha-1-\varepsilon},\ 0 < \varepsilon < \alpha - 1, & \text{if } \alpha \leqslant 3/2,\ v' = \alpha - \varepsilon, \\ \text{const}\, r & \text{if } \alpha > 3/2,\ v' = 2-\alpha, \\ \text{const}\, r^{\alpha} & \text{if } \alpha > 3/2,\ v' > 1, \\ \text{const}\, r^2 & \text{if } \alpha > 3/2,\ v' = 3-\alpha \end{cases}$$

(the bounds come mainly from one of the terms of I_1, called I_{15}), and

$$|I_2| \leqslant \begin{cases} \text{const}\, r & \text{if } v' = 2-\alpha, \\ \text{const}\, r^{\alpha-1+v'} & \text{if } \alpha \leqslant 3/2,\ v' < \alpha, \\ & \text{or } \alpha > 3/2,\ \alpha - 1 < v' < 3-\alpha, \\ \text{const}\, r^{2\alpha-1-\varepsilon},\ 0 < \varepsilon < 1, & \text{if } \alpha \leqslant 3/2,\ v' = \alpha, \\ \text{const}\, r^2 & \text{if } \alpha > 3/2,\ v' = 3-\alpha. \end{cases}$$

Since $E[X_2^2|X_1 = 0] < \infty$, Theorem 1.1 implies that (1.2) (or (1.1)) holds with $v' = 2 - \alpha$. Using the preceding bounds, we get $|I_i| \leqslant \text{const}\, r$ for $i = 1,2$ and therefore (3.1) always holds if $v < 3 - \alpha$. If we consider $v \geqslant 3 - \alpha$, then, of course, we may assume that (1.2) holds with all $v' < 3 - \alpha$. Thus, for $\alpha \leqslant 3/2$, $|I_1| + |I_2| < \text{const}\, r^{2\alpha-1-\varepsilon}$ with arbitrarily small $\varepsilon > 0$ since $\alpha \leqslant 3 - \alpha$. This implies (3.1) for all $v < \alpha + 1$ and Theorem 1.2 is proved for $\alpha \leqslant 3/2$. In the case $3/2 < \alpha < 2$, we may need to use the above chart where the bounds for I_1 and I_2 are doubled or tripled (depending on the value of v). □

4. Technical results

This section contains three elementary lemmas which are used in the proofs of Theorems 1.1 and 1.2.

The first lemma is an easy consequence of the definition of the Gamma function. It is included here because it is often referred to in the proofs.

Lemma 4.1. *For* $c > 0$, $\alpha > 0$, $\beta > -1$

$$\int_0^\infty e^{-c^\alpha r^\alpha} r^\beta \, dr = \Gamma\left(\frac{\beta+1}{\alpha}\right) \Big/ (\alpha c^{\beta+1}). \tag{4.1}$$

The following result appears as Lemma 3.5 in (1994c) (cf. also (2.6) in (1994b)).

Lemma 4.2. *For* $z \in \mathbb{R}$ *and* $0 < \beta < 3$,

$$|(z+1)^{\langle\beta\rangle} + (z-1)^{\langle\beta\rangle} - 2z^{\langle\beta\rangle}| \leqslant \text{const} \min(|z|^{\min(1,\beta)}, |z|^{\beta-2}),$$

where const *depends only on* β.

The last lemma, which determines the sign of $(z+1)^{\langle\beta\rangle} + (z-1)^{\langle\beta\rangle} - 2z^{\langle\beta\rangle}$, is the crucial one.

Lemma 4.3. *For* $z \geqslant 0$ *and* $0 < \beta < 1$

$$(z+1)^\beta + (z-1)^{\langle\beta\rangle} - 2z^\beta \leqslant 0$$

and for $1 < \beta < 3$

$$(z+1)^\beta + (z-1)^{\langle\beta\rangle} - 2z^\beta \geqslant 0.$$

Moreover, for $1 < \beta < 3$, *there exists* $0 < c(\beta) < 1/2$ *such that for* $0 < z < c(\beta)$

$$(z+1)^\beta + (z-1)^{\langle\beta\rangle} - 2z^\beta \geqslant \text{const}\, z, \tag{4.2}$$

where const *depends only on* β.

Proof. Assume $1 < \beta < 3$. First note that if $z \geqslant 1$ then by convexity of the power function we have

$$(z+1)^\beta + (z-1)^{\langle\beta\rangle} - 2z^\beta = (z+1)^\beta + (z-1)^\beta - 2z^\beta$$
$$\geqslant 2\left(\frac{z+1+z-1}{2}\right)^\beta - 2z^\beta = 0.$$

Now, assume $0 \leqslant z \leqslant 1$ and put

$$f(z) := (z+1)^\beta + (z-1)^{\langle\beta\rangle} - 2z^\beta = (z+1)^\beta + (1-z)^\beta - 2z^\beta.$$

Then

$$f'(z) = \beta[(z+1)^{\beta-1} + (1-z)^{\beta-1} - 2z^{\beta-1}]$$

and

$$f''(z) = \beta(\beta-1)[(z+1)^{\beta-2} - (1-z)^{\beta-2} - 2z^{\beta-2}].$$

Again using the convexity of a power function with exponent greater or equal to one, we get $f'(z) \geqslant 0$ for $2 \leqslant \beta < 3$. Thus, in this case $f(z) \geqslant f(0) = 0$.

If $1 < \beta < 2$, then $f''(z) < 0$, so that $f'(z)$ is decreasing in $(0, 1)$. But $f'(0^+) = 2\beta > 0$ and $f'(1^-) = \beta(2^{\beta-1} - 2) < 0$, which implies that f has exactly one maximum in $(0, 1)$. Thus, $f(z) \geqslant \min(f(0), f(1)) = 0$.

The inequality for $0 < \beta < 1$ can be proved similarly. Use the concavity of the power function for $z \geqslant 1$ and show that $f(z)$, for $0 \leqslant z \leqslant 1$, has exactly one minimum.

To prove (4.2) note that for $0 < z < 1/2$

$$
\begin{aligned}
&(z + 1)^\beta - (1 - z)^\beta - 2z^\beta \\
&= 1 + \beta z + \beta(\beta - 1)(1 + \theta_1 z)^{\beta-2}\frac{z^2}{2} - 1 + \beta z - \beta(\beta - 1)(1 - \theta_2 z)^{\beta-2}\frac{z^2}{2} - 2z^\beta \\
&\geqslant \begin{cases} 2\beta z - 2z^\beta & \text{if } 2 \leqslant \beta < 3, \\ 2\beta z + \beta(\beta - 1)((\frac{2}{3})^{2-\beta} - 2^{2-\beta})\frac{z^2}{2} - 2z^\beta & \text{if } 1 < \beta < 2. \end{cases}
\end{aligned}
$$

In both cases the right-hand side of the above inequality is greater than const z for sufficiently small z, i.e. for $z < c(\beta)$. □

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